Mathematical Foundation Needed for PMML Engineering Courses

The Manufacturing Engineering courses in the PMML program are developed at the Master's level. As such, while they are intended to be more practical than theoretical, graduate level Engineering mathematics are required and expected at certain points in the curriculum. Especially, we expect that students will have a "mastery" level of familiarity with the concepts listed below from Calculus and Probability and Statistics.

Students who have not seen this material before, or who have not practiced it in many years, are strongly encouraged to pursue some of the online resources provided for each topic prior to starting the PMML program and/or as needed to address deficiencies that become evident while taking courses. We have ordered the topics in the sequence that we recommend for reviewing/going through them. As reviewing/learning the material, please remember that in mathematics comprehending an instructor/online video while they are lecturing/teaching is just the first step. The ability to use and master the mathematics requires practice. There is no short cut/substitute for this practice.

<u>It is recommended that all PMML students review **Section IV**.</u>

We recommend LinkedIn Learning (www.linkedinlearning.com) for all topics.

Also, Khan Academy (www.khanacademy.org) has modules that cover most topics. You will need to register for an account with Khan Academy, but all of the information is provided free of charge.

I. Precalculus

Concepts from precalculus are required to master the mathematical skills given in Sections II. – V. Some past PMML students have found it necessary to complete an online course and/or to brush up in precalculus before moving into the Calculus and/or Probability and Statistics material. If you feel comfortable with your Precalculus preparation/experience, you can skip this topic in your preparation. Else, to start out verifying your mathematics preparation by reviewing your precalculus skills in Khan Academy (link above).

II. Differentiation (Calculus)

Differentiation is required in homework problems and exams throughout the curriculum. Following are some commonly used formulae that will be useful as you go through the PMML courses. If you need to learn to use these formulae, we recommend the resources that follow.

Basic Properties/Formulas/Rules

$\frac{d}{dx}\left(cf(x)\right) = c\frac{df(x)}{dx}$, c is any constant

$\frac{d}{dx}\left(f(x) \pm g(x)\right) = \frac{df(x)}{dx} \pm \frac{dg(x)}{dx}$

$\frac{d}{dx}(x^n) = nx^{n-1}$, n is any number

$\frac{d}{dx}(c) = 0$, c is any constant

Product Rule: $\frac{d}{dx}\left(f(x)g(x)\right) = \left(\frac{df(x)}{dx}\right)g(x) + f(x)\left(\frac{dg(x)}{dx}\right)$

Quotient Rule: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\left(\frac{df(x)}{dx}\right)g(x) - f(x)\left(\frac{dg(x)}{dx}\right)}{g(x)^2}$

Chain Rule: $\frac{d}{dx}\left(f\left(g(x)\right)\right) = \frac{df\left(g(x)\right)}{dx}\left(\frac{dg(x)}{dx}\right)$

$$\frac{d}{dx}(\ln g(x)) = \frac{1}{g(x)}\frac{dg(x)}{dx}$$

Common Derivatives

Polynomials

$$\frac{d}{dx}(cx^n) = ncx^{n-1}$$

Exponential/Logarithm Functions

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}, x > 0$$

Learning Resources

Calculus 1A: Differentiation

Discover the derivative---what it is, how to compute it, and when to apply it in solving real world problems. Part 1 of 3.

III. Integration (Calculus)

Integration is also required in homework problems and exams throughout the curriculum. Following are some commonly used formulae that will be useful as you go through the PMML courses. If you need to learn to use these formulae, we recommend the resources that follow.

<u>Basic Properties/Formulas/Rules</u>

$\int cf(x)dx = c\int f(x)dx$, c is a constant $\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$

$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$ where $F(x) = \int f(x)dx$

$\int_a^b cf(x)dx = c\int_a^b f(x)dx$, c is a constant $\int_a^b f(x) \pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

$\int_a^a f(x)dx = 0$ $\int_a^b f(x)dx = -\int_b^a f(x)dx$

$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ $\int_a^b cdx = c(b-a)$

If $f(x) \geq 0$ on $a \leq x \leq b$ then $\int_a^b f(x)dx \geq 0$

If $f(x) \geq g(x)$ on $a \leq x \leq b$ then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$

<u>Common Integrals</u>

Polynomials

$\int dx = x + c$ $n \neq -1$

$\int \frac{1}{x}dx = \ln|x| + c$ $n \neq 1$

Exponential/Logarithm Functions

$$\int \ln u\, du = u \ln u - u + c$$

$\int ue^u du = (u-1)e^u + c$

<u>Standard Integration Techniques</u>

Substitution

Given $\int_a^b f\big(g(x)\big)\left(\frac{dg(x)}{dx}\right)dx$ then the substitution $u = g(x)$ will convert this into the integral,

$\int_a^b f\big(g(x)\big)\left(\frac{dg(x)}{dx}\right)dx = \int_{g(a)}^{g(b)} f(u)du$

Integration by Parts

The standard formulas for integration by parts are,

$\int udv = uv - \int vdu$ $f(t)|_a^b = f(b) - f(a)$

Choose u and dv and then compute du by differentiating u and compute v

Learning Resources

Calculus 1B: Integration
Discover the integral - what it is and how to compute it. See how to use calculus to model real world phenomena. Part 2 of 3.

IV. Probability and Statistics

Undoubtedly, probability and statistics is the largest area where PMML students find that they need to put in additional time/preparation as they go through the curriculum. Even recent graduates with Engineering degrees often find that their undergraduate Prob-Stats preparation was lacking relative to what is needed for this graduate program. It is recommended that all incoming PMML students review at least some of this material.

Application of Calculus to Probability

Let T be a continuous random variable. T is specified by its cumulative distribution function (CDF) $F(t)$. For example, the CDF for the exponential distribution is

$$F(t) = 1 - \exp(-\lambda t)\,, t \geq 0$$

Note that λ is a parameter of the distribution and the domain of the random variable is all t greater or equal to 0. The probability density function (PDF) of T is

$$f(t) = \frac{dF(t)}{dt}$$

The PDF has the property that $0 \leq \int_a^b f(t)dt \leq 1$ and $\int_{-\infty}^{\infty} f(t)dt = 1$. For the exponential distribution, the PDF is:

$$f(t) = \frac{d}{dt}(1 - \exp(-\lambda t)) = \lambda \exp(-\lambda t)$$

The expectation of T, denoted $\mathbb{E}[T]$, is a measure of central tendency for the distribution. The variance, $Var(T)$ is a measure of its dispersion. The variance is the expectation of the squared deviation of T from the mean: $Var(T) = \mathbb{E}[(T - \mathbb{E}[T])^2] = \mathbb{E}[T^2] - \mathbb{E}[T]^2$.

Expectation:

$$\mathbb{E}[T] = \int_{-\infty}^{\infty} t f(t) dt$$

Variance:

$$Var(T) = \int_{-\infty}^{\infty} t^2 f(t) dt - \mathbb{E}[T]^2$$

For the exponential distribution:

$$\mathbb{E}[T] = \int_0^{\infty} \lambda t \exp(-\lambda t)\, dt$$

This can be solved using uv-substitution. Let $u = t$ and $dv = \lambda \exp(-\lambda t)\, dt$. Then $du = dt$ and $v = -\exp\,(-\lambda t)$. Now we apply $uv - \int vdu$:

$$\mathbb{E}[T] = -t\exp(-\lambda t)\,|_0^\infty - \int_0^\infty -\exp(-\lambda t)\,dt = \int_0^\infty \exp(-\lambda t)\,dt$$

The first term vanishes because $\exp(-\lambda t)$ approaches zero as $t \to \infty$ faster than t approaches ∞ as $t \to \infty$. To simplify calculations note that $\frac{\lambda}{\lambda}\mathbb{E}[T] = \mathbb{E}[T]$. Therefore,

$$\mathbb{E}[T] = \frac{1}{\lambda}\int_0^\infty \lambda\exp(-\lambda t)\,dt = \frac{1}{\lambda}$$

since the integral of the PDF of T is one.

We can use a similar trick to compute the variance.

$$Var(T) = \int_0^\infty \lambda t^2\exp(-\lambda t)\,dt - \mathbb{E}[T]^2$$

To solve the integral, let $u = t^2$ and $dv = \lambda\exp(-\lambda t)\,dt$. Therefore, $du = 2tdt$ and $v = -\exp(-\lambda t)$. Now we apply $uv - \int vdu$:

$$Var(T) = \left[-t^2\exp(-\lambda t)\,|_0^\infty - \int_0^\infty -2t\exp(-\lambda t)\,dt\right] - \frac{1}{\lambda^2}$$

The first term vanishes because $\exp(-\lambda t)$ approaches zero as $t \to \infty$ faster than t^2 approaches ∞ as $t \to \infty$. Furthermore, $\mathbb{E}[T]^2 = \frac{1}{\lambda^2}$. Now to solve the integral $\int_0^\infty 2t\exp(-\lambda t)\,dt$, note that it is equivalent to $\frac{2}{\lambda}\int_0^\infty \lambda t\exp(-\lambda t)\,dt = \frac{2}{\lambda}\mathbb{E}[T] = \frac{2}{\lambda^2}$. Therefore, $Var(T) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$.

Learning Resources

1. Probability and Statistics GT online course ISYE6739
2. LinkedIn Learning (www.linkedinlearning.com)
   The two recommended sessions on probability and statistics are in the Business Intelligence section: Statistics with Excel Part One and Part Two.
3. LinkedIn Learning, Lynda Courses

Statistics Foundations: The Basics
Statistics Foundations: Probability
Statistics Foundations: 2
Statistics Foundations: 3

4. Amazon.com: Probability & Statistics Tutor: Random Variables & Introduction to Probability Distributions: Jason Gibson:

   Amazon Digital Services LLC

V. Linear Differential Equations

Mastery of Differential Equations is not generally required to complete the PMML curriculum. Knowledge of basic principles of Diff EQ, however, will help students in following the online lectures and understanding how/why formulae are derived.

$$\frac{dy}{dt} + p(t)y = g(t) \tag{1}$$

Multiply both sides by an unknown integrating factor $\mu(t)$

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)g(t) \tag{2}$$

Assume $\mu(t)$ satisfies the following:

$$\mu(t)p(t) = \frac{d\mu(t)}{dt} \tag{3}$$

Substitute (3) into (2)

$$\mu(t)\frac{dy}{dt} + \frac{d\mu(t)}{dt}y = \mu(t)g(t) \tag{4}$$

This is the product rule:

$$\frac{d}{dt}\big(\mu(t)y(t)\big) = \mu(t)g(t) \tag{5}$$

Multiply both sides by dt and integrate:

$$\int \frac{d}{dt}\big(\mu(t)y(t)\big)dt = \int \mu(t)g(t)dt$$

$$\mu(t)y(t) + c = \int \mu(t)g(t)dt \tag{6}$$

Using algebra, we arrive at the solution:

$$y(t) = \frac{\int \mu(t)g(t)dt + c}{\mu(t)} \tag{7}$$

Note the sign of the constant is irrelevant. To find $\mu(t)$, refer to (3).

$$\mu(t)p(t) = \frac{d\mu(t)}{dt}$$

Multiply both sides by $\frac{1}{\mu(t)}dt$

$$p(t)dt = \frac{d\mu(t)}{\mu(t)} \tag{8}$$

Integrate both sides:

$$\int p(t)dt = \int \frac{1}{\mu(t)}d\mu(t) = \ln \mu(t) + k \tag{9}$$

Move the constant of integrality to the other side (sign does not matter) and take exponential of both sides:

$$\mu(t) = \exp\left(\int p(t)dt + k\right) = K\exp\left(\int p(t)dt\right) \tag{10}$$

where $K = \exp k$
Solution to a linear first order differential equation is:

$$y(t) = \frac{\int \mu(t)g(t)dt + c}{\mu(t)}$$

$$\mu(t) = K \exp\left(\int p(t)dt\right) \tag{11}$$

Example: Let $g(t) = \lambda_1 \exp(-(\lambda_1 + \lambda_2)t)$. Find the solution for

$$\frac{dy}{dt} + \lambda_2 y = g(t)$$

Where initial conditions are $g(0) = 1$, $y(0) = 0$
Solution:

$$\mu(t) = K \exp\left(\int \lambda_2 dt\right) = K \exp(\lambda_2 t)$$

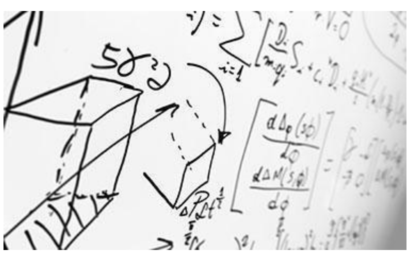
$$y(t) = \frac{\lambda_1 \int \exp(-\lambda_1 t)\, dt + c}{\exp(\lambda_2 t)}$$

$$\lambda_1 \int \exp(-\lambda_1 t)\, dt = -\exp(-\lambda_2 t)$$

$$y(t) = -\exp(-(\lambda_1 + \lambda_2)t) + c \exp(-\lambda_1 t)$$

$$y(0) = -1 + c = 0, c = 1$$

Learning Resources



Differential Equations
Differential equations are the language of the models we use to describe the world around us.